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Thermodynamics of entanglement in Schwarzschild spacetime

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Abstract

Extending the analysis in our previous paper, we construct the entanglement thermodynamics for a massless scalar field on the Schwarzschild spacetime. Contrary to the flat case, the entanglement energy E_{ent} turns out to be proportional to area radius of the boundary if it is near the horizon. This peculiar behavior of E_{ent} can be understood by the red-shift effect caused by the curved background. Combined with the behavior of the entanglement entropy, this result yields, quite surprisingly, the entanglement thermodynamics of the same structure as the black hole thermodynamics. On the basis of these results, we discuss the relevance of the concept of entanglement as the microscopic origin of the black hole thermodynamics.

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I. INTRODUCTION

One of the central problems in black hole physics is the identification of the microscopic origin of black hole entropy, which obeys the relation

$$S_{BH} = \frac{1}{4l_{Pl}^2}\mathcal{A}, \tag{1.1}$$

where $\mathcal{A}$ is the area of the event horizon and $l_{\rm Pl} := \sqrt{G}$ is the Planck length [1]. (Hereafter we will set $c = \hbar = 1$.) There are two important facts which suggest that the black hole entropy may have some microscopic origin. One is Hawking's argument [2] on quantum fields in a black hole spacetime showing that the black hole emits thermal radiation with the temperature T_{BH} which is proportional to that determined by the classical 1st law for black holes [3] with a universal coefficient. In fact, the coefficient on the right-hand side of Eq.(1.1) is chosen so that T_{BH} coincides with the latter. For derivation of T_{BH}, see [4,5] as well as [2]. The other is the fact that the free energy calculated from the Euclidean path integral for the pure gravity system gives in the saddle point approximation exactly the same expressions for the temperature and the entropy as those given above. On the basis of these facts and their consistency, there have been proposed various candidates for the microscopic origin of the black hole entropy [6]. Among them the simplest one is the idea of the entanglement entropy [7,8], which focuses on the entropy associated with states of quantum fields on black hole spacetimes and is strongly motivated by the first of the above two facts.

The entanglement entropy itself is quite a general concept which is nothing but coarse graining entropy for a quantum system caused by an observer's partial ignorance of the information on the state. Now the idea is that the event horizon may play the role of the boundary of perception for an asymptotic observer so that its existence may give rise to the entropy. Indeed for simple models on a flat spacetime [7,8] explicit estimations showed that the entanglement entropy S_{ent} is always proportional to the area $\mathcal{A}$ of the boundary between two regions of a spatial section of spacetime:

$$S_{ent} = \frac{C_S}{a^2}\mathcal{A}, \tag{1.2}$$

where a is a cut-off scale for regularization and C_S is a model-dependent coefficient of order unity. Thus there is a clear similarity between S_{ent} and S_{BH}.

In order to see whether there is something deeper in this similarity, we have constructed a kind of thermodynamics of a space boundary, which we call entanglement thermodynamics, for a massless scalar field on flat spacetime in our previous paper [9]. There, by giving a suitable definition of the entanglement energy E_{ent}, we determined the entanglement temperature T_{ent} by imposing the first law of thermodynamics. Then we compared the system of thermodynamics quantities $(E_{ent}, S_{ent}, T_{ent})$ obtained by this procedure with the corresponding one for black hole, $(E_{BH} = M, S_{BH}, T_{BH})$. This comparison showed that the entanglement thermodynamics on a flat background possesses a totally different structure compared with the black hole one.

It is not difficult to understood why such discrepancy occurs. Since we wanted to construct thermodynamics of a space boundary, we have defined the entanglement energy E_{ent} so that it depends only on quantum degree of freedom around the space boundary. As a result, E_{ent} became proportional to the boundary area $\mathcal{A}$ unlike the black hole energy $E_{BH} = M \propto \sqrt{\mathcal{A}}$.

Since the dependence of the thermodynamical quantities on the boundary area come from the very nature of the idea of entanglement, it may appear that the idea of entanglement cannot have any relevance to the black hole thermodynamics beyond the similarity of the expressions for entropy. However, it is not the case. It is because gravity is not taken into account in the above argument. In fact there is a good reason to expect that the inclusion of gravity improves the situation drastically.

In black hole spacetime the energy of quantum field gets gravitational corrections, which depend on the definition of the energy. Taking into account of the fact that E_{ent} depends only on modes around the boundary, the entanglement energy estimated on flat background, $E_{ent,\mathrm{flat}}$, should be identified with the energy $E_{ent,\mathrm{b}}$ measured by a Killing observer located

near the horizon. On the other hand, since the black hole mass used as the energy in the black hole thermodynamics corresponds to the energy measured by an observer at spatial infinity, it is natural to use the corresponding quantity $E_{ent,\infty}$. Due to the gravitational redshift, these energies are related by

$$E_{ent,\infty} = (-g_{tt})_{\mathrm{b}}^{1/2} E_{ent,\mathrm{b}} \simeq (-g_{tt})_{\mathrm{b}}^{1/2} E_{ent,\mathrm{flat}}, \tag{1.3}$$

where $(-g_{tt})_{\mathrm{b}}^{1/2}$ is the well-known red-shift factor [1] for a signal emitted at the boundary.

Let us assume that the boundary is at the proper distance $a \sim l_{\mathrm{Pl}}$ from the horizon. Then for the Schwarzschild metric we obtain $(-g_{tt})_{\mathrm{b}}^{1/2} \simeq a/(2M) \propto 1/\sqrt{\mathcal{A}}$. Hence from $E_{ent,\mathrm{flat}} \propto \mathcal{A}$, if follows that $E_{ent,\infty} \propto \sqrt{\mathcal{A}}$. On the other hand, S_{ent} is independent of the position of an observer once a quantum state is fixed, so $S_{ent} \propto \mathcal{A}$ as before. Thus it is expected that the gravity effect modifies the structure of the entanglement thermodynamics so that it becomes identical to that of the black hole thermodynamics.

On the basis of these observations, in this paper we construct the entanglement thermodynamics for a massless scalar field on the Schwarzschild background, and compare its structure with that of the black hole thermodynamics.

The paper is organized as follows. In the next section we describe the model used in the paper in detail, and clarify the definitions of the entanglement entropy and energy and the basic assumptions with brief explanations of their motivations. Then, after explaining our regularization scheme, we derive general formulas for the entanglement entropy and energy of the regularized system and estimate them numerically in §3 and §4. On the basis of these results, we compare the structure of the entanglement thermodynamics for the massless scalar field in the Schwarzschild background with that of the black hole in §5.

II. MODEL CONSTRUCTION

In order to construct the entanglement thermodynamics of a black hole, we must make clear how to implement the idea of entanglement into a quantum field system on a black

hole spacetime, and give the definitions of the basic thermodynamical quantities in terms of the quantum fields.

A. Model description

The basic idea of the entanglement thermodynamics is to express the thermodynamical quantities for a black hole in terms of expectation values of quantum operators dependent on the spacetime division as in the statistical mechanics modeling of the thermodynamics for ordinary systems. Therefore we must specify how to divide spacetime into two regions and with respect to what kind of state the expectation values are taken.

According to the original idea of entanglement, it is clearly most natural to consider a dynamical spacetime describing black hole formation from a nearly flat spacetime in the past infinity, and divide the spacetime into the regions inside and outside the horizon. In this situation, if we start from the asymptotic Minkowski vacuum in the past, the entanglement entropy associated with the division of spacetime by the horizon acquires a clear physical meaning. However, this ideal modeling cause difficulties. The most serious one is caused by the occurrence of the Hawking radiation: its contribution to the entanglement entropy diverges. Of course, if the backreaction effect is properly taken into account, this divergence may disappear. But such a modeling is intractable at present and is beyond the scope of the present paper.

In order to avoid this difficulty, we replace the problem by a stationary one on the basis of the following arguments. First, consider a situation in which a thin spherical shell with a tiny mass δM infalls toward the origin to form a Schwarzschild black hole with mass δM. In this situation, the timelike Killing surface Σ with $r = 2G\Delta M$ in the Minkowski region turns to the horizon after the passage of the shell. Two things happen in this process: a division of spacetime by the horizon and a change of geometry. Among these two things, the latter is responsible for the Hawking radition, because the Hawking radiation is essentially determined by the geometry near the horizon in the geometrical optics approximation.

Meanwhile, the division of spacetime produces the entropy of the black hole in our approach based on the entanglement. Thus, we can determine the entropy of the black hole avoiding the above difficulty caused by the Hawking radition, if we calculate the entanglement entropy associated with the division of Minkowski spacetime by the timelike surface Σ starting from the Minkowski vacuum. Because we need to introduce a cut-off length a of the order of the Planck length l_{Pl}, a natural minimum value of ΔM is determined by the condition $r^2 \sim a^2$. The corresponding value of the entropy becomes of order unity.

Next, let us consider a thin spherical shell with a tiny mass ΔM infalling toward the center in a Schwarzschild background spacetime with mass M. After the passege of the mass shell, a Schwarzschild black hole with mass $M + \Delta M$ is formed. As in the previous case, this process produces a new horizon which divides the spacetime by a null surface corresponding to the stationary timelike surface Σ with $r = 2G(M + \delta M)$ before the passege of the shell, and slightly modifies the geometry. If we apply the reasoning in the previous case to the present situation, we can expect that the entropy of the new black hole is given by the entanglement entropy for quantum fields in the Killing vacuum state, associated with the division of the Schwarzschild spacetime by the timelike surface Σ before the passage of the shell. Here, the Killing vacuum state should be taken in order to avoid the Hawking radiation. Now the minimum value of ΔM is determined by the condition that the increase of the area of the horizon is of order a^2, which corresponds to an increase of entropy of order unity. For this minimum value, the proper distance of Σ to the original horizon becomes of order a if $a \sim l_{Pl}$.

On the basis of these arguments, in this paper, we consider a quantum field in the Killing vacuum state in the Schwarzschild background with mass M,

$$ds^2 = -(1 - r_0/r)dt^2 + (1 - r_0/r)^{-1}dr^2 + r^2 d\Omega^2, \tag{2.1}$$

where $r_0 = 2GM$, and calculate the entanglement entropy and the entanglement energy to be defined later associated with the division of the spacetime by a timelike surface Σ at a proper distance of the order of a cut-off length($\sim l_{Pl}$) to the horizon. Since there is

no definite criterion on the exact position of the boundary in our framework, we will also investigate the influence of the variation of its position.

Here a subtlety occurs: if we require that the quantum field is in the Killing vacuum state on the whole extended Kruscal spacetime and take the horizon as the boundary surface, then the entanglement entropy vanishes because the state is expressed as the tensor product of the Killing states in the regions I and II (*Figure* 1). This triviality comes from the fact that the quantum degree of freedom on the horizon is completely eliminated for such a state. In order to avoid this, we restrict the spacetime into the region I, and replace the boundary by a time-like surface Σ at a proper distance of the order of the cut-off length (typically, Planck length) of the theory to the horizon [1]. The boundary Σ can be regarded as a would-be horizon when matter fields (a real scalar field in our analysis in this paper), which contribute to the entanglement entropy, falls into the black hole. Evidently, the would-be horizon must be very close to the horizon since our analysis is semi-classical one. However, because of the existence of the cut-off length there is the minimum proper length between the would-be and the present horizon: it must be of order of the cut-off length. Hence it is natural to take as the boundary a timelike surface at a proper distance of the order of the cut-off length to the horizon. This prescription is expected to give a correct estimate of the thermodynamical quantities for the case in which the spacetime is divided by the horizon and the quantum state inside the horizon is arbitrary specified because the entanglement quantities depend only on the degrees of freedom within the distance of the order of the cut-off length from the boundary. However, we should keep in mind that we have no definite criterion regarding the exact position of the boundary. To minimize this ambiguity, we will also investigate the influence of the variation of the boundary position.

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[1] Our present model clears such a criticism [10] for works done before [7–9,11] that in these works the boundary had always been chosen as a timelike surface, contrary to the event horizon, which is a null surface.

As a matter content we consider a real scalar field described by

$$S = -\frac{1}{2}\int \{\partial^\mu \phi \partial_\mu \phi + \mu^2\}\sqrt{-g}dx^4 \quad . \tag{2.2}$$

The mass μ does not play an essential role since a typical length scale controlling the entanglement thermodynamics is much smaller than the Compton length of a usual field. Therefore we just set $\mu = 0$ in the numerical computations.

B. Entanglement entropy and energy

There is no difficulty in generalizing the entanglement entropy for the flat case [7–9] to the case of Schwarzshild background. In contrast, with regards to the definition of the entanglement energy, we cannot simply extend the definition used for the flat case [9] due to the difference in the method of implementation of the idea of entanglement as well as due to the existence of the black hole mass in the present case.

Before discussing the definition of the entanglement energy, let us first recall the standard general procedure of building a statistical mechanics model for thermodynamics. Let A be a subsystem surrounded by the environment B. For a quasi-static process, the first law of thermodynamics for the subsystem A is expressed as

$$dE = TdS + fdx \quad . \tag{2.3}$$

Here E, S and T are the thermodynamical energy, entropy and temperature of A, respectively, f is a generalized force applied to A, and x is a generalized displacement. Now the statistical mechanics model of this thermodynamics is constructed by the following sequence of steps:

(a) Assign a microscopic model to A and construct a Hamiltonian system for it.

(b) Make statistical assumptions (e.g. the principle of equi-weight) which define averaging procedures for the microscopic variables.

(c) Identify the thermodynamical variables with quantities constructed from the microscopic variables introduced in (a) along with the averaging procedure in (b).

(d) Reproduce the relations between the thermodynamical variables (thermodynamical laws in particular).

Since the basic idea of the entanglement thermodynamics is to construct a statistical mechanics model of the black hole thermodynamics using the idea of entanglement, we should also follow a similar procedure to this general one in defining E_{ent}.

In our case it is natural to regard the scalar field in the region inside the boundary Σ as the subsystem A, and the scalar field in the region outside Σ as the environment B. Then, if we decompose the Hamiltonian H corresponding to the Killing energy to the parts dependent on the microscopic degrees of freedom inside Σ, those outside Σ and both as $H = H_{in} + H_{out} + H_{int}$, the most natural choice of the microscopic variable for E_{ent} is the operator H_{in}. Clearly the averaging procedure is given by taking the expectation value with respect to the Killing vacuum of the total Hamiltonian H. Hence we are led to the definition

(I) $E^{(I)}_{ent} = \langle : H_{in} : \rangle$,

where $: H_{in} :$ denotes the normal ordering with respect to the ground state of H_{in}.

Here note that the choice

(I') $E^{(I')}_{ent} = \langle : H_{out} : \rangle$

is essentially same as (I) in spite of its appearance, as we will see in Sec.IV. It is because E_{ent} measures a sort of disturbance to the vacuum state caused by the boundary Σ, and its value is determined just by the modes around Σ. Hence the difference between (I) and (I') comes from a tiny difference in the redshift effect on the modes just inside Σ and just outside Σ.

These choices should be contrasted with the definition of the energy in the standard argument of black hole thermodynamics. There the energy is given by the black hole mass

M, which is quite natural in the framework in which the backreaction of matter on gravity is consistently taken into account. However, in the present case, the option

(II) $E_{ent}^{(II)} = M$,

lacks charm in our framework because it is not related to microscopic degrees of freedom. Of course, it may be reasonable to replace the definition (I) by

(III) $E_{ent}^{(III)} = M + \langle : H_{in} : \rangle$,

in the sense that it represents a kind of total energy of the system consisting of the scalar field and the gravitational field.

Here we should comment on a subtlety of the role of M in our model. Since we are neglecting the backreaction of the matter on gravity, the black hole mass M determining the background geometry g_{ab} might be regarded as the external parameter x in the above expression for the 1st law of thermodynamics. However, since the Killing vacuum state depends on M as we will see later, M becomes a function of thermodynamical variables S_{ent} and E_{ent}. Thus in effect the term $f dx$ in Eq.(2.3) does not appear in the present case [12].[2]

Because there is no firm ground to pick up one of these, we will investigate all of them. Further, for the sake of comparison with the result in the flat spacetime models discussed in our previous paper [9], we also calculate the quantities defined by

(IV_1) $E_{ent}^{(IV_1)} := \langle : H : \rangle_{\rho'}$,

(IV_2) $E_{ent}^{(IV_2)} = \langle : H_{in} : \rangle + \langle : H_{out} : \rangle$.

Here $: H :$ and $: H_{out}$ denote the normal products with respect to the ground states of H and H_{out}, respectively, and $\langle \cdot \rangle_{\rho'}$ indicates the expectation value with respect to the density

[2] It is not a surprise that an external parameter becomes a function of variables describing thermodynamics state. For example, in the Rindler spacetime, the acceleration α is determined by the Rindler temperature.

matrix $\rho' := \rho_{in} \otimes \rho_{out}$, where ρ_{in} and ρ_{out} are the reduced density matrices for the inside and the outside of Σ, respectively [9].

Finally we comment on the regularization. In dealing with the matter field, we naturally encounter the ultraviolet divergence. We adopt here the cut-off regularization by introducing the length scale a, which is supposed to be of order of $l_{\rm Pl}$. On dimensional grounds we expect that $S_{ent} \propto \mathcal{A}/a^2$ and $E_{ent} \propto \sqrt{A}/a^2$. Thus the entanglement temperature $T_{ent} = dE_{ent}/dS_{ent}$ is expected to be independent of the cut-off scale a. We will see later that this is the case except for the definitions (II) and (III).

III. ENTANGLEMENT ENTROPY

A. Basic formulas

In order to estimate the entanglement entropy and the entanglement energy for the scalar field on the Schwarzschild spacetime, we first regularize the field theory described by Eq.(2.2) and reduce it to a discrete canonical system described by a Hamiltonian of the following form :

$$H_0 = \sum_{A,B=1}^{N} \frac{1}{2a} \delta^{AB} p_A p_B + \frac{1}{2} V_{AB} q^A q^B, \tag{3.1}$$

where $\{(q^A, p_A)\}$ $(A = 1, 2, \cdots, N)$ are canonical pairs. Here both q^A's and p_A's are of physical dimension $[L^0]$, and the parameter a is the cut-off length of order of $l_{\rm Pl}$.

Because of the spherical symmetry of the system, if we expand the scalar field ϕ in terms of the real spherical harmonics as

$$\phi(t, \rho, \theta, \varphi) = \sum_{l,m} \phi_{lm}(t, \rho) Z_{lm}(\theta, \varphi), \tag{3.2}$$

the action becomes a simple sum of the contributions from each mode ϕ_{lm}. Here $Z_{lm} = \sqrt{2}\Re Y_{lm}$, $\sqrt{2}\Im Y_{lm}$ for $m > 0$ and $m < 0$, respectively, with $Z_{l0} = Y_{l0}$, and ρ is a suitable radial coordinate. Hence if we discretize this radial coordinate, we obtain a regularized system.

As the radial coordinate, we adopt the proper length from the horizon, which is related to the circumferential radius coordinate r by

$$\rho = \int_{r_0}^{r} \frac{dr}{\sqrt{1-\frac{r_0}{r}}} = r_0 \left\{ \frac{\nu}{1-\nu^2} + \ln \frac{1+\nu}{\sqrt{1-\nu^2}} \right\}, \tag{3.3}$$

where $\nu := \sqrt{1-\frac{r_0}{r}}$. In this coordinate the metric Eq.(2.1) is written as

$$ds^2 = -\nu^2 dt^2 + d\rho^2 + r_0^2 \frac{d\Omega^2}{(1-\nu^2)^2}, \tag{3.4}$$

where ν is understood as a function of ρ through Eq.(3.3).

Plugging Eqs.(3.4) and (3.2) into Eq.(2.2), we get

$$S = \frac{1}{2} \sum_{lm} \int dt\ d\rho \left[\frac{r_0^2 \dot{\phi}_{lm}^2}{\nu(1-\nu^2)^2} - \left\{ \frac{r_0^2 \nu}{(1-\nu^2)^2} \left(\partial_\rho \phi_{lm}\right)^2 + \left(l(l+1) + \frac{(r_0\mu)^2}{(1-\nu^2)^2} \right) \nu \phi_{lm}^2 \right\} \right]. \tag{3.5}$$

Comparing this equation with Eq.(3.1) we see that the most suitable configuration variable is the dimensionless one defined by

$$\psi_{lm}(t,\rho) := \frac{r_0}{\nu^{1/2}(1-\nu^2)} \phi_{lm}(t,\rho). \tag{3.6}$$

Hereafter we adopt a boundary condition that $\psi_{lm}(t,\rho)$ is finite at the horizon, which corresponds to the Dirichlet boundary condition for the original variable $\phi_{lm}(t,\rho)$. To be precise, in defining Hamiltonian for our system, which must be self-adjoint, there are two options for the boundary condition of $\phi_{lm}(t,\rho)$ at the horizon: the Dirichlet boundary condition and the Neumann boundary condition. In our case we adopt the former since the later makes the Killing energy divergent.

Now it is straightforward to perform the canonical transformation. The Poisson bracket relations become

$$\begin{aligned} &\{\psi_{lm}(\rho), \pi_{l'm'}(\rho')\} = \delta_{ll'}\delta_{mm'}\delta(\rho-\rho') \quad , \\ &\{\psi_{lm}(\rho), \psi_{l'm'}(\rho')\} = 0 \quad , \\ &\{\pi_{lm}(\rho), \pi_{l'm'}(\rho')\} = 0 \quad , \end{aligned} \tag{3.7}$$

where π_{lm} is a momentum conjugate to ψ_{lm} with dimension $[L^{-1}]$. In terms of these canonical quantities the Hamiltonian is expressed as

$$
\begin{aligned}
H &= \sum_{lm} H_{lm} \quad , \\
H_{lm} &= \frac{1}{2}\int d\rho \pi_{lm}^2(\rho) + \frac{1}{2}\int d\rho d\rho' \psi_{lm}(\rho) V_{lm}(\rho,\rho')\psi_{lm}(\rho'),
\end{aligned} \tag{3.8}
$$

where

$$
\begin{aligned}
\psi_{lm}(\rho)V_{lm}(\rho,\rho')\psi_{lm}(\rho') = \delta(\rho,\rho')\Big[&\frac{\nu}{(1-\nu^2)^2}\Big\{\partial_\rho\left(\nu^{1/2}(1-\nu^2)\psi_{lm}\right)\Big\}^2 \\
&+\frac{\nu^2}{r_0^2}\left\{l(l+1)(1-\nu^2)^2+(r_0\mu)^2\right\}\psi_{lm}^2\Big].
\end{aligned} \tag{3.9}
$$

We regularize this system by replacing it by a difference system with respect to ρ with spacing a. To be precise, we make the following replacements:

$$
\begin{aligned}
\rho &\to Aa, \\
\psi_{lm}(\rho=(A-1/2)a) &\to q_{lm}^A, \\
\pi_{lm}(\rho=(A-1/2)a) &\to p_{lmA}, \\
\nu(\rho=(A-1/2)a) &\to \nu_A, \\
\delta(\rho=Aa,\rho=Ba) &\to \delta_{AB}/a,
\end{aligned}
$$

where A runs over the positive integers. To achieve a better precision, we adopt the middle-point prescription in discretizing the terms including a derivative: we replace a term, say, $f(\rho)\partial_\rho g(\rho)$ by $f_{A+1/2}\cdot\frac{1}{a}(g_{A+1}-g_A)$. Further, in order to make the degrees of the system finite, we impose the boundary conditions $q_{lm}^{N+1}=0$. In the numerical calculation N is taken to be sufficiently large so that this artificial boundary condition, which is required just for a technical reason, does not affect the results.

In this manner we get the hamiltonian in the desired form:

$$
\begin{aligned}
H_0 &= \sum_{lm} H_0^{(lm)} \quad , \\
H_0^{(lm)} &= \sum_{A,B=1}^{N}\left[\frac{1}{2a}\delta^{AB}p_{lmA}p_{lmB}+\frac{1}{2}V_{AB}^{(lm)}q_{lm}^A q_{lm}^B\right] \quad ,
\end{aligned} \tag{3.10}
$$

where

$$\sum_{A,B=1}^{N} V_{AB}^{(lm)} q_{lm}^A q_{lm}^B = \frac{1}{a}\sum_{A=1}^{N}\left[\frac{\nu_{A+1/2}}{(1-\nu_{A+1/2}^2)^2}\left(\nu_{A+1}^{1/2}(1-\nu_{A+1}^2)q_{lm}^{A+1} - \nu_A^{1/2}(1-\nu_A^2)q_{lm}^A\right)^2 \right.$$
$$\left. + \left(\frac{a}{r_0}\right)^2 \nu_A^2 \left(l(l+1)(1-\nu_A^2)^2 + (r_0\mu)^2\right) q_{lm}^{A\ 2}\right].$$

Here $V^{(lm)}$ becomes the positive definite, symmetric matrix whose explicit form is

$$\left(V_{AB}^{(lm)}\right) = \frac{2a}{r_0^2}\begin{pmatrix} \Sigma_1^{(l)} & \Delta_1 & & & & \\ \Delta_1 & \Sigma_2^{(l)} & \Delta_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \Delta_{A-1} & \Sigma_A^{(l)} & \Delta_A & \\ & & & \ddots & \ddots & \ddots \end{pmatrix},$$

$$\Sigma_A^{(l)} = \frac{r_0^2}{2a^2}\nu_A(1-\nu_A^2)^2\left(\frac{\nu_{A+1/2}}{(1-\nu_{A+1/2}^2)^2} + \frac{\nu_{A-1/2}}{(1-\nu_{A-1/2}^2)^2}\right)$$
$$+\frac{1}{2}\nu_A^2\left(l(l+1)(1-\nu_A^2)^2 + (r_0\mu)^2\right),$$
$$\Delta_A = -\frac{r_0^2}{2a^2}\nu_{A+1/2}(\nu_{A+1}\nu_A)^{1/2}\frac{(1-\nu_{A+1}^2)(1-\nu_A^2)}{(1-\nu_{A+1/2}^2)^2}\ . \qquad (3.11)$$

Here note that the mass term in $\Sigma_A^{(l)}$ is negligible compared with the first term if $a\mu << 1$. Therefore we will simply set $\mu = 0$ in the numerical calculations.

B. Formulas for the entanglement entropy

In the discretized system[3] $\left((q^A, p_A); H_0\right)$ $(A = 1, 2, \cdots, N)$ with Eq.(3.10), the position of the space boundary Σ is given by $\rho = \rho_B := n_B a$ with $n_B = O(1)$, which the set of modes $\{q^A\}$ $(A = 1, \cdots, N)$ into two subsets, $\{q^a\}$ $(a = 1, \cdots, n_B)$ and $\{q^\alpha\}$ $(\alpha = n_B + 1, \cdots, N)$. Here we regard $\{q^a\}$ and $\{q^\alpha\}$ as the inside modes and the outside modes, respectively.

The Killing vacuum for the continuous system corresponds to the ground state of the Hamiltonian H_0 in this system. Hence its density matrix is given by

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[3] For the notational simplicity we will often omit the suffices (l, m) if no confusion occurs.

$$\rho(\{q^A\},\{q'^B\}) = \left(\det\frac{W}{\pi}\right)^{1/2}\exp\left[-\frac{1}{2}W_{AB}(q^Aq^B+q'^Aq'^B)\right] \quad , \tag{3.12}$$

where $W = (aV)^{1/2}$. In accordance with the splitting of $\{q^A\}$ into $\{q^a\}$ and $\{q^\alpha\}$, the matrices V, W and its inverse W^{-1} naturally split into four blocks as

$$\begin{aligned}(V)_{AB} &= \begin{pmatrix} V^{(1)}_{ab} & (V_{int})_{a\beta} \\ ({V_{int}}^T)_{\alpha b} & V^{(2)}_{\alpha\beta}\end{pmatrix}, \\ (W)_{AB} &= \begin{pmatrix} A_{ab} & B_{a\beta} \\ (B^T)_{\alpha b} & D_{\alpha\beta}\end{pmatrix}, \\ \left(W^{-1}\right)_{AB} &= \begin{pmatrix} \tilde{A}^{ab} & \tilde{B}^{a\beta} \\ (\tilde{B}^T)^{\alpha b} & \tilde{D}^{\alpha\beta}\end{pmatrix}.\end{aligned} \tag{3.13}$$

Taking the partial trace of $\rho(\{q^A\},\{q'^B\})$ for the inside modes $\{q^a\}$ ($a = 1,\cdots,n_B$), we get the reduced density matrix

$$\begin{aligned}\rho_{\rm red}(\{q^\alpha\},\{q'^\beta\}) = \left(\det\pi\tilde{D}\right)^{-1/2}\exp\Big[&-\frac{1}{2}(\tilde{D}^{-1})_{\alpha\beta}(q^\alpha q^\beta + q'^\alpha q'^\beta) \\ &-\frac{1}{4}(B^TA^{-1}B)_{\alpha\beta}(q-q')^\alpha(q-q')^\beta\Big].\end{aligned}$$

Now the entanglement entropy associated with the boundary Σ, $S_{ent} := -{\rm Tr}\rho_{\rm red}\ln\rho_{\rm red}$, is given as follows [7,8]. Let $\{\lambda_i\}$ ($i = 1,\cdots,N-n_B$) be the eigenvalues of a positive definite symmetric matrix[4] Λ,

$$\Lambda := \tilde{D}^{1/2}B^TA^{-1}B\tilde{D}^{1/2}. \tag{3.14}$$

Then it is easily shown that modes labeled by (l,m) contribute to the entangle entropy by the amount

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[4] The corresponding expression in ref. [7] ("$\Lambda^a{}_b := (M^{-1})^{ac}N_{cb}$") reads $\Lambda = \tilde{D}B^TA^{-1}B$ in the present notation. This definition does not give a symmetric matrix and should be replaced by "$\Lambda^{ab} := (M^{-1/2})^{ac}N_{cd}(M^{-1/2})^{db}$", namely Eq.(3.14).

$$
\begin{aligned}
S_{ent}^{(l)} &= \sum_{i=1}^{N-n_B} S_i, \\
S_i &= -\frac{\mu_i}{1-\mu_i} \ln \mu_i - \ln(1-\mu_i),
\end{aligned} \tag{3.15}
$$

where $\mu_i := \lambda_i^{-1}\left(\sqrt{1+\lambda_i}-1\right)^2$. (Note that $0 < \mu_i < 1$.) Clearly $S_{ent}^{(l)}$ is independent of $m = (-l, -l+1, \cdots, l-1, l)$. The entanglement entropy is given by

$$
S_{ent} = \sum_{l=0}^{\infty}(2l+1)S_{ent}^{(l)}. \tag{3.16}
$$

From Eq.(3.11), one can easily show that

$$
S_{ent}^{(l)} \sim O\left((la/r_0)^{-4}\ln(la/r_0)\right) \quad \text{as} \quad la/r_0 \to \infty.
$$

Thus the infinite series Eq.(3.15) actually converge so that we can safely truncate them at some appropriate l, depending on the accuracy we require and the ratio r_0/a we set.

C. Numerical estimation

Using these formulas, we have evaluated S_{ent} numerically and have examined its dependence on the area of the boundary, $\mathcal{A} = 4\pi r_B^2$. In this calculation the outer numerical boundary is set at $N = 60$. The summation in l in Eq.(3.16) is taken has up to $l = [10r_0/a]$ ([ ] is the Gauss symbol). From the above asymptotic behavior of $S_{ent}^{(l)}$, this guarantees the accuracy of 10%.

The result is shown in *Figure* 2. From this figure we see that S_{ent} is proportional to $\mathcal{A}/a^2$ if we change r_0 with fixed n_B, and its coefficient only has a weak dependence on n_B. In particular, for the limit $n_B = 1$, we get

$$
S_{ent} \simeq 0.024\mathcal{A}/a^2 \quad . \tag{3.17}
$$

This result is essentially the same as our previous result for models in a flat spacetime including the numerical coefficient [9]. This can be understood in the following way.

Let us make a coordinate change from r to x defined by

$$\frac{r}{r_0} = \frac{(x+1)^2}{4x},$$

or

$$x = \frac{2r}{r_0} - 1 + \sqrt{\left(\frac{2r}{r_0} - 1\right)^2 - 1}.$$

Then Eq.(2.1) is rewritten as

$$ds^2 = -\left(\frac{x-1}{x+1}\right)^2 dt^2 + r_0^2 \left(\frac{1+x}{2x}\right)^4 (dx^2 + x^2 d\Omega^2).$$

Note that $r = 0$, r_0 and ∞ correspond to $x = 0$, 1 and ∞, respectively. It is easy to see that the Hamiltonian (Eq.(3.8)) is given in this coordinate system as

$$H = \sum_{lm} H_{lm},$$
$$H_{lm} = \int d\xi \frac{64x^4(x-1)}{(x+1)^7} \left[\frac{1}{2}P_{lm}^2 + \frac{1}{2}\left(\frac{(x+1)^2}{4x}\right)^4 \left\{(\partial_\xi \varphi_{lm})^2 + \frac{l(l+1)}{\xi^2}\varphi_{lm}^2\right\}\right], \tag{3.18}$$

where $\xi := r_0 x$, and P_{lm} and φ_{lm} are expressed as $P_{lm} := \frac{r_0}{64}\frac{(x+1)^7}{x^4(x-1)}\dot{\phi}_{lm}$ and $\varphi_{lm} := r_0 \phi_{lm}$ in terms of ϕ_{lm} in Eq.(3.2). Here note that the vacuum state is only weakly dependent on the prefactor $\frac{64x^4(x-1)}{(x+1)^7}$ in Eq.(3.18). If we neglect this prefactor, the vacuum state is determined by the Hamiltonian which coincides with that for the flat spacetime at $x = 1$. On the other hand, S_{ent} depends on the modes in a thin layer around the boundary Σ, whose typical thickness is a few times of $a \simeq l_{\rm Pl}$). Therefore, when Σ is near the horizon, the value of S_{ent} should be well approximated the flat spacetime value.

IV. ENTANGLEMENT ENERGY

In this section we give formulas for the various definitions of the entanglement energy introduced in Sec.II B, and estimate their values numerically.

First we derive formulas for the entanglement energies corresponding to the definitions (I) and (I'):

$$E_{ent}^{(I)} := \langle : H_{in} : \rangle, \tag{4.1}$$

$$E_{ent}^{(I')} := \langle : H_{out} : \rangle. \tag{4.2}$$

By rescaling the variables $\left\{ q^A \right\}$ in §III as

$$\bar{q}^A := \delta^{AB} \left(W^{1/2} \right)_{BC} q^C,$$

the expression of the density matrix for the vacuum state Eq.(3.12) gets simplified as

$$\langle \left\{ \bar{q}^A \right\} | \rho | \left\{ \bar{q}'^B \right\} \rangle = \prod_{C=1}^{N} \pi^{-1/2} \exp \left[-\frac{1}{2} \left\{ (\bar{q}^C)^2 + (\bar{q}'^C)^2 \right\} \right],$$

and the normal ordered Hamiltonian $: H_{in} :$ is represented as

$$\begin{aligned} : H_{in} : &= -\frac{1}{2a} \delta^{ab} \left(\frac{\partial}{\partial q^a} - w_{ac}^{(1)} q^c \right) \left(\frac{\partial}{\partial q^b} + w_{bd}^{(1)} q^d \right) \\ &= -\frac{1}{2a} U^{AB} \left(\frac{\partial}{\partial \bar{q}^A} - \bar{w}_{AC}^{(1)} \bar{q}^C \right) \left(\frac{\partial}{\partial \bar{q}^B} + \bar{w}_{BD}^{(1)} \bar{q}^D \right). \end{aligned}$$

Here $w^{(1)}$ is the positive square-root of $aV^{(1)}$, and the matrices U and $\bar{w}^{(1)}$ are defined as

$$\begin{aligned} U^{AB} &:= \delta^{AC} \left(W^{1/2} \right)_{Ca} \delta^{ab} \left(W^{1/2} \right)_{bD} \delta^{DB}, \\ \bar{w}_{AB}^{(1)} &:= \delta_{AC} \left(W^{-1/2} \right)^{Ca} w_{ab}^{(1)} \left(W^{-1/2} \right)^{bD} \delta_{DB}. \end{aligned}$$

Hence the matrix elements of $: H_{in} : \rho$ with respect to the basis $|\bar{q}^A\rangle$ are expressed as

$$\begin{aligned} \langle \left\{ \bar{q}^A \right\} | : H_{in} : \rho | \left\{ \bar{q}^B \right\} \rangle = &\frac{1}{2a} \left\{ \left[(\bar{w}^{(1)} + 1) U (\bar{w}^{(1)} - 1) \right]_{AB} \bar{q}^A \bar{q}^B + \mathrm{Tr} \left[U (1 - \bar{w}^{(1)}) \right] \right\} \\ &\times \prod_{C=1}^{N} \pi^{-1/2} \exp \left[-(\bar{q}^C)^2 \right]. \end{aligned}$$

From this we obtain[5]

$$\begin{aligned} E_{ent}^{(I)} &= \int \left(\prod_{C=1}^{N} d\bar{q}^C \right) \langle \left\{ \bar{q}^A \right\} | : H_{in} : \rho | \left\{ \bar{q}^B \right\} \rangle \\ &= \frac{1}{4a} \left[a V_{ab}^{(1)} (\tilde{A})^{ab} + A_{ab} \delta^{ab} - 2 w_{ab}^{(1)} \delta^{ab} \right]. \end{aligned} \tag{4.3}$$

---

[5] See Eq.(3.13) for the definitions of the matrices A, $\tilde{A}$, D and $\tilde{D}$.

Similarly $E_{ent}^{(I')}$ is expressed as

$$E_{ent}^{(I')} = \frac{1}{4a}\left[aV^{(2)}_{\alpha\beta}(\tilde{D})^{\alpha\beta} + D_{\alpha\beta}\delta^{\alpha\beta} - 2w^{(2)}_{\alpha\beta}\delta^{\alpha\beta}\right], \tag{4.4}$$

where $w^{(2)}$ is the positive square-root of $aV^{(2)}$.

E_{ent} corresponding to the definition (III) are simply related to $E_{ent}^{(I)}$ by

$$E_{ent}^{(III)} = M + E_{ent}^{(I)} \tag{4.5}$$

Further, E_{ent} corresponding to the definitions (IV) have already been given for the flat case [9]. They are expressed as

$$\begin{aligned} E_{ent}^{(IV_1)} &:= \mathrm{Tr}\left[: H_{tot} : \rho'\right], \\ E_{ent}^{(IV_2)} &:= \mathrm{Tr}\left[(: H_{in} : + : H_{out} :)\rho\right]. \end{aligned} \tag{4.6}$$

Here $H_{tot} := H_{in} + H_{out} + H_{int}$ and $\rho' := \rho_{in} \otimes \rho_{out}$ [9].[6] In particular for a vacuum state, they become [9]

$$\begin{aligned} E_{ent}^{(IV_1)} &= -\frac{1}{2}\mathrm{Tr}\left[V_{int}^T \tilde{B}\right] \\ E_{ent}^{(IV_2)} &= -\frac{1}{2}\mathrm{Tr}\left[V_{int}^T \tilde{B}\right] - \frac{1}{2a}\left(\mathrm{Tr}\left[(aV^{(1)})^{1/2}\right] + \mathrm{Tr}\left[(aV^{(2)})^{1/2}\right] - \mathrm{Tr}\left[W\right]\right), \end{aligned} \tag{4.7}$$

where V_{int}, $V^{(1,2)}$ and W are given in Eq.(3.13).

Like the entanglement entropy (Eq.(3.16)), the entanglement energy is also given by the summation of each contribution specified by (l, m):

$$E_{ent} = \sum_{l=0}^{\infty}(2l+1)E_{ent}^{(l)}. \tag{4.8}$$

Here E_{ent} represents any kind of the entanglement energy defined above (except for $E_{ent}^{(II)} = M$ and $E_{ent}^{(III)} = M + E_{ent}^{(I)}$) and $E_{ent}^{(l)}$ is its (l, m)-contribution. From Eq.(3.11), it is easily shown that

---

[6]Note that $E_{ent}^{(IV_1)}$ and $E_{ent}^{(IV_2)}$ here are the generalizations to the curved background case of E_{ent}^{I} and E_{ent}^{II}, respectively, in Ref. [9]. In the same way the subscripts '*in*' and '*out*' correspond to '1' and '2' in Ref. [9].

$$E^{(l)}_{ent} \sim O\left((la/r_0)^{-3}\right) \quad \text{as} \quad la/r_0 \to \infty.$$

Thus the summation with respect to l in (4.8) converges.

With the helps of these formulas, we have numerically evaluated $E^{(I)}_{ent}$, $E^{(I')}_{ent}$ $E^{(IV_1)}_{ent}$ and $E^{(IV_2)}_{ent}$. Now we have taken the numerical outer boundary at $N = 100$. The truncation in the l-summation is the same as for S_{ent} (up to $l = [10r_0/a]$), which implies that the accuracy is about 10% from the above asymptotic estimate for $E^{(l)}_{ent}$.

The results of numerical calculations are shown in *Figure* 3, *Figure* 4, *Figure* 5 and *Figure* 6. In these figures, $r_B E_{ent}$ is plotted as a function of $(r_B/a)^2$ for $n_B = 1, 2, 5$. All of these figures show that $r_B E_{ent}$ is proportional to $n_B(r_B/a)^2$:

$$\begin{aligned}
E^{(I)}_{ent} &\sim 0.05(n_B - 1/2)r_B/a^2, \\
E^{(I')}_{ent} &\sim 0.05(n_B + 1/2)r_B/a^2, \\
E^{(IV_1)}_{ent} &\sim 0.2 n_B r_B/a^2, \\
E^{(IV_2)}_{ent} &\sim 0.1 n_B r_B/a^2.
\end{aligned} \tag{4.9}$$

Note that the identity $E^{(IV_2)}_{ent} = E^{(I)}_{ent} + E^{(I')}_{ent}$ holds from the definitions.

From these equations we immediately see that the values of $E^{(I)}_{ent}$ and $E^{(I')}_{ent}$ coincide except for a tiny difference independent of n_B. As was mentioned in Sec.II B, this difference is understood by the gravitational red-shift: $E^{(I)}_{ent}$ comes from the modes just inside Σ while $E^{(I')}_{ent}$ originates from the modes just outside Σ. In the present numerical calculations, it means that $E^{(I)}_{ent}$ and $E^{(I')}_{ent}$ are determined by the modes at $\rho = (n_B - 1/2)a$ and $\rho = (n_B + 1/2)a$, respectively(see Eq.(3.11)). Hence, taking account of the fact that the contribution of each mode to the entanglement energy is proportional to the red-shift factor at its location, the ratio of $E^{(I)}_{ent}$ and $E^{(I')}_{ent}$ should be approximately given by

$$\begin{aligned}
E^{(I)}_{ent} : E^{(I')}_{ent} &\sim \nu(\rho = (n_B - 1/2)a) : \nu(\rho = (n_B + 1/2)a) \\
&\sim (n_B - 1/2) : (n_B + 1/2).
\end{aligned} \tag{4.10}$$

This is consistent with the above numerical result.

This argument is also supported by the numerical result for the flat spacetime model shown in *Figure* 7. In this figure the values of $E_{ent}^{(I)}$ and $E_{ent}^{(I')}$ for a massless scalar field in the Minkowski spacetime with $\Sigma = S^2$ are plotted. In this case there is no gravitational red-shift effect, so we expect that $E_{ent}^{(I)} = E_{ent}^{(I')}$, as confirmed by the numerical calculation.

V. COMPARISON BETWEEN ENTANGLEMENT THERMODYNAMICS AND BLACK-HOLE THERMODYNAMICS

Now on the basis of our results for S_{ent} and E_{ent}, let us compare the structure of the entanglement thermodynamics with that of black hole.

From the numerical results in §3 and §4, the entanglement entropy S_{ent} and the entanglement energy E_{ent} for our model are expressed as

$$S_{ent} = C_S \left(\frac{l_{\rm Pl}}{a}\right)^2 \frac{\mathcal{A}}{4l_{\rm Pl}^2} \quad (C_S \simeq 0.096), \tag{5.1}$$

$$E_{ent} = C_E \frac{r_B}{2a^2} = C_E \left(\frac{r_B}{r_0}\right) \left(\frac{l_{\rm Pl}}{a}\right)^2 Mc^2, \tag{5.2}$$

where

$$\begin{aligned}
C_E^{(I)} &\simeq 0.05, \quad C_E^{(I')} \simeq 0.15, \\
C_E^{(II)} &= \left(\frac{r_0}{r_B}\right)\left(\frac{a}{l_{\rm Pl}}\right)^2 \simeq \left(\frac{a}{l_{\rm Pl}}\right)^2, \\
C_E^{(III)} &= \left(\frac{r_0}{r_B}\right)\left(\frac{a}{l_{\rm Pl}}\right)^2 + C_{E^{(I)}} \simeq \left(\frac{a}{l_{\rm Pl}}\right)^2 + C_{E^{(I)}}, \\
C_E^{(IV_1)} &\simeq 0.4, \quad C_E^{(IV_2)} \simeq 0.2.
\end{aligned} \tag{5.3}$$

Here and hereafter we will only consider the horizon limit, $n_B = 1$.

It is helpful to keep in mind that the case $C_S = C_E = 1$ along with $a = l_{\rm Pl}$ corresponds to the black hole thermodynamics.[7]

---

[7] Strictly speaking $\mathcal{A}$ in Eq.(5.2) is the area of the boundary and it differs from the area of the

From these expressions and the first law of thermodynamics

$$dE = TdS \tag{5.4}$$

the entanglement temperature T_{ent} is determined as

$$T_{ent} = \frac{C_E}{C_S} T_{BH}. \tag{5.5}$$

Thus we get

$$\begin{aligned}
T_{ent}^{(I)} &\simeq 0.52 T_{BH}, \quad T_{ent}^{(I')} \simeq 1.6 T_{BH} \quad , \\
T_{ent}^{(II)} &\simeq \left(\frac{a}{l_{\rm Pl}}\right)^2 T_{BH} \quad , \\
T_{ent}^{(III)} &\simeq \left\{ \left(\frac{a}{l_{\rm Pl}}\right)^2 + 0.52 \right\} T_{BH} \quad , \\
T_{ent}^{(IV_1)} &\simeq 4.2 T_{BH}, \quad T_{ent}^{(IV_2)} \simeq 2.1 T_{BH}.
\end{aligned} \tag{5.6}$$

These results have several interesting features. First of all we immediately see that the entanglement thermodynamics on the Schwarzschild spacetime show exactly the same behavior as the black hole thermodynamics, as summarized in *Table* I. This behavior is just what we expected from the intuitive argument in the introduction: the gravitational redshift effect modifies the area dependence of E_{ent} so as to make the entanglement thermodynamics behave just like the black hole thermodynamics.

Second it should be noted that the temperature T_{ent} becomes independent of the cut-off scale a only for the options type (I) and (IV) (i.e. (I), (I'), (IV_1) and (IV_2)). This indicates that (II) and (III) are not good definitions of the energy in our approach. It is also suggestive that the average $\frac{1}{2}(T_{ent}^{(I)} + T_{ent}^{(I')})$ gives almost the same value as T_{BH}. This averaging corresponds to averaging out the difference in the red-shift factors for the one-mesh 'inside' and the one-mesh 'outside' of the boundary. Therefore such an averaging may have some meaning.

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horizon. However we here ignore the difference since it is totally negligible as compared with $\mathcal{A}$ itself. Similarly $\frac{r_B - r_0}{r_0} = O\left((\frac{a}{r_0})^2\right)$ so that we can safely set $r_B \simeq r_0$ in the present context.

To summarize, our model analysis strongly suggests a tight connection between the entanglement thermodynamics and the black hole thermodynamics. It is worth emphasizing that matter fields on a black hole spacetime capture the background characteristics. Of course, our model is too simple to give any definite conclusion based on it. In particular, the ambiguity in the definition of the energy comes from neglecting backreaction of the quantum field on gravity. Further, even in the fixed background framework, our model is too simple in that its thermodynamics is essentially controlled by one parameter corresponding to the black hole mass. It is obviously useful to see whether the entanglement thermodynamics is consistent with the black hole thermodynamics for models with more parameters, such as those on the Reisner-Nordstrom spacetime or on the Kerr spacetime before attacking the difficult task of going beyond the semi-classical approximation.

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FIGURES

FIG. 1. The Kruskal extension of the Schwarzschild spacetime. We consider only the region I (the shaded region). As the boundary Σ we take the hypersurface $r = r_B$.

FIG. 2. The numerical evaluations for S_{ent} for the discretized theory of the scalar field in Schwarzschild spacetime. S_{ent} for $n_B = 1, 2, 5$ is shown as functions of A/a^2, where A is area of the boundary. We have taken $N = 100$ and performed the summation over l up to $10r_0/a$.

FIG. 3. The numerical evaluations for $E_{ent}^{(I)}$ for the discretized theory of the scalar field in Schwarzschild spacetime. $r_B E_{ent}^{(I)}$ for $n_B = 1, 2, 5$ is shown as functions of $(r_B/a)^2$, where $r_B \equiv r(\rho = n_B a)$. We have taken $N = 100$ and performed the summation over l up to $10r_0/a$.

FIG. 4. The numerical evaluations for $E_{ent}^{(I')}$ for the discretized theory of the scalar field in Schwarzschild spacetime. $r_B E_{ent}^{(I')}$ for $n_B = 1, 2, 5$ is shown as functions of $(r_B/a)^2$, where $r_B \equiv r(\rho = n_B a)$. We have taken $N = 100$ and performed the summation over l up to $10r_0/a$.

FIG. 5. The numerical evaluations for $E_{ent}^{(IV_1)}$ for the discretized theory of the scalar field in Schwarzschild spacetime. $r_B E_{ent}^{(IV_1)}$ for $n_B = 1, 2, 5$ is shown as functions of $(r_B/a)^2$, where $r_B \equiv r(\rho = n_B a)$. We have taken $N = 100$ and performed the summation over l up to $10r_0/a$.

FIG. 6. The numerical evaluations for $E_{ent}^{(IV_2)}$. $r_B E_{ent}^{(IV_2)}$ for $n_B = 1, 2, 5$ is shown as functions of $(r_B/a)^2$, where $r_B \equiv r(\rho = n_B a)$. We have taken $N = 100$ and performed the summation over l up to $10r_0/a$.

FIG. 7. The numerical evaluations for $E_{ent}^{(I)}$ and $E_{ent}^{(I')}$ in the Minkowski spacetime. $aE_{ent}^{(I)}$ and $aE_{ent}^{(I')}$ are shown as functions of $(r_B/a)^2$, where r_B is the radius of the sphere which provides the division of region. We have taken $N = 60$ and performed the summation over l up to $10r_0/a$.

TABLES

TABLE I. Comparison of two kinds of thermodynamics

	Entanglement in Schwarzschild spacetime	*Black-Hole*
Varied	A	A
Fixed	a	l_{Pl}
S	$\propto A$	$\propto A$
E	$\propto A^{1/2}$	$\propto A^{1/2}$
T	$\propto A^{-1/2}$	$\propto A^{-1/2}$
Varied	a	l_{Pl}
Fixed	A	A
S	$\propto a^{-2}$	$\propto l_{\mathrm{Pl}}^{-2}$
E	$\propto a^{-2}$	$\propto l_{\mathrm{Pl}}^{-2}$
T	$\propto a^{0}$	$\propto l_{\mathrm{Pl}}^{0}$
Varied	a	l_{Pl}
Fixed	E_{ent}	M
S	$\propto a^{2}$	$\propto l_{\mathrm{Pl}}^{2}$
E	$\propto a^{0}$	$\propto l_{\mathrm{Pl}}^{0}$
T	$\propto a^{-2}$	$\propto l_{\mathrm{Pl}}^{-2}$